CS522 Computational Finance

TREASURIES

Characteristics, Markets, Databases, Smoothing

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Chapter 1

Government Debt

All over the world governments finance their activities by issuing government obligations (also called "sovereign debt") in local currency, or in the currency of a foreign country.

Government obligations are often backed by certain guarantees of repayment, which are meant to alleviate investors' worries; thus, to reduce the risk premium embedded in the price the respective governments can charge for their obligations. Holders of a given country's sovereign debt frequently enjoy tax advantages in the respective country.

In countries where the independence of the autochthonous central bank is weak, governments have the option of printing money to inflate away their debt denominated in local currency. Investors, especially at the institutional level, are of course aware of this possibility, and they charge substantial risk premia on such obligations. If the debt is issued in the currency of another country, then inflating it away is not an option; if a government is unable to service its debt denominated in foreign currencies, it has the option to default. Indeed, country defaults are not rare events. In recent years, the default on Russia's sovereign debt, and on that of Argentina, has rocked the financial world. Country defaults do not typically erase debts, however. Because the amounts involved are large, and because the typical lender is a large institutional investor or another government, country defaults are followed by protracted negotiations to restructure and/or reschedule the debt, or otherwise change the contractual obligations underlying the defaulted debt. Often, these negotiations end in a significant reduction of the net present value of the debt. Still, default has devastating consequences for a country's economy, especially because the country is subsequently shut out of the international financial market for a long period of time. Perceptions about a country's creditworthiness are expressed by international country ratings, and are meant to inform investors' decision to acquire debt issued by a particular government.

Major industrial countries are considered to be very low risk debt issuers, among them the United States, whose rise to the pinnacle of financial reliability has paralleled, and has been helped by its emergence as the major political and military factor in the world. As an aside, we note that as late as the 19th century the US had several hundred banks issuing currency on the basis of authorizations obtained from state legislatures. At the time the American presence on the international financial markets was marginal at best. The establishment of the Fed and the acceptance of a "federal" currency were major advances in US Financial history.

American government debt is backed "by the full faith of the United States Government."

The US government issues debt for two principal reasons:

(i) To finance its deficit;

(ii) To manage the time mismatch between its income (principally: taxes) and expenditures.

The US government has been running deficits for decades, and - after a brief interruption at the end of the 1990s - this trend is projected to continue indefinitely in the future. Given the size of the American economy, and the size of the deficit, this all but assures that a huge amount of US debt will be available on the financial markets.

1.1 Types and Main Characteristics of US Government Obligations

American government debt is typically classified based on its at-issue maturity (the time when the debt comes due) into bills, notes and bonds. Instruments with original maturity up to one year are called Treasury bills ("T-bills"), those with an original maturity between 1 and 10 years are Treasury notes ("T-notes"), and those with maturities over 10 years are Treasury bonds ("T-bonds").

Treasuries are characterized by their "face value," which is the amount that the US Treasury will pay when the respective instrument comes due; this amount is also known as "principal." In addition to this, notes and bonds pay interest in the form of coupons. If a note or a bond pays a coupon of C (typically specified per $100 face value), then the investor will receive an amount of $(face\ value) * \frac{C}{200}$ twice a year, at six-month intervals (in other words, the yearly coupon is paid out in two equal parts). A coupon is paid at the time the bond or note comes due; the other payment times are defined - roughly speaking - by counting back 6-month intervals from the time of maturity.

Treasuries, bills, notes, and bonds, are held by a very large and diverse collection of investors, private, institutional, and governmental. In recent years, some of the largest holders of US debt have been the Chinese, Japanese, and Korean governments.[1] Because of the safety and predictability of Treasuries, they are included in the portfolios of insurance companies and pension funds. Treasuries are also used to hedge risk exposure in other instruments. Low net-worth[2] individuals

[1]This is done primarily to prevent the appreciation of the main East-Asian currencies; thus, to maintain the competitiveness of US imports originating from this area.

[2]We refer here to the lower tier of investors who actually have enough funds to

often prefer Treasuries because of safety (there is no $100,000-guarantee limit on Treasuries, for example, as opposed to bank deposits), but also because they offer a slightly higher interest rate than the bank deposit rates that are accessible to retail small investors.

Treasuries are exempt from certain state and local taxes, and as such, they are used for tax planning purposes by both individuals and institutional investors. Not all institutional investors are interested in Treasuries' tax advantages, however; pension funds holding 401(k) funds, for example, are mostly exempt from taxes.

Treasuries are identified based on their CUSIP codes. CUSIPs are unique identifiers assigned to various securities, including Government and corporate bonds, by the Committee on Uniform Securities Identification Procedures. Certain Treasury data providers, like the University of Chicago's Center for Research in Security Prices (CRSP), define alternative unique identifiers for the securities they report on.[3] There is a one-to-one correspondence between these alternative identifiers and CUSIP codes.

1.1.1 Unusual Treasuries

In the past, the US has issued Treasuries that were "callable," i.e. the government had the right to redeem these obligations before they would mature. No such instruments are issued as of now. Other unusual instruments have been issued in the past, including the so-called "flower bonds," which offered inheritance-tax advantages to those holding them at the time of their death. Such unusual features

worry, for example, about the $100,000 bank guarantee limit. As such, a low net worth investor could still be reasonably wealthy compared to the average population.

[3] In the case of CRSP, these alternative identifiers are the CRSP ids.

make bonds harder to value.

Starting in 1997, the US started to sell "Treasury Inflation Protected Securities," also known as "TIPS." TIPS pay coupons, but their principal is not constant, rather it is adjusted by indexing to the non-seasonally adjusted U.S. city average all items Consumer Price Index for all Urban Consumers. Indexing is lagged (by 3-months in the US). While they still represent a relatively low percentage of total US debt, TIPS are considered revolutionary because they allow the mitigation of inflation risk. TIPS are not completely free of inflation risk: one component of this residual risk is due to the lag in following the inflation index, the second component is due to the imprecisions in the definition of the index itself. Like regular Treasuries, TIPS are exempt from local and state taxes, and under certain conditions, they are also exempt from federal taxes (Sundaresan 2002, pp. 250).

The principal and coupon payment structure of notes and bonds, similarly to other market convention oddities, is due to historical legacy. In the past a principal reason for coupon payments was primarily to provide retail investors with constant, almost continuous flow of income during the life of the bond.[4] From the perspective of today's typical investor - which is an institution - this feature is not important; using derivatives these entities can restructure the timing of their future cash flows almost at will. As we will argue when we compute the forward rate curve, the packaging of coupon and principal payments together creates great practical difficulties. These difficulties are widely recognized, and they led to the emergence of the so-called "stripped" Treasuries, also known as STRIPS ("Separate Trading of Interest and Principal Securities"). Stripping a Treasury means that its associated coupon and principal payments are separated, and they trade

[4]And also - maybe - to provide constant reassurance about the ability of the debtor to pay.

independently. STRIPS are, in this sense, the long term equivalent of bills. Not all Treasuries are eligible for stripping. Representing a single payment at a well defined moment in the future, strips have properties that are advantageous especially for hedging long term obligations (for example, their duration[5] is always equal to their leftover maturity). As such, strips are often in high demand, which can lead to price distortions (Sundaresan 2002, pp. 230). Because of the documented price-distorting effects of strip demand, stripped securities are typically not included in the data set when computing forward rate curves.

1.2 Issuance of Treasuries

Treasuries are issued by the US Treasury at regular intervals, at dates and times fixed and announced well in advance. These schedules have been changed from time to time in the past, but they remain constant over long periods of time. The set of bond maturities that are issued changes over time. For example, the Treasury has decided to suspend the issuance of 30-year bonds beginning in October 2001 (Bureau of the Public Debt 2004).

Treasuries can also be sold, in addition to regularly scheduled sales, by issuing so-called cash-management bills. These are instruments that mature at the same time as an already outstanding issue of bills (Cook and LaRoche 1993, pp. 75). The Treasury occasionally redeems certain debt instruments on the open market. Calls are also not unusual.[6] All these actions change the total outstanding amount

[5]"Duration" is a measure of a bond's sensitivity to changes in interest rates (parallel shifts of the forward rate curve) (Jarrow and Turnbull 1999). While only of limited applicability, it is widely used. Simple duration hedging is achieved when the duration of assets matches the duration of liabilities.

[6]At the time of this writing, the latest bond call issued by the Treasury Department was that of a bond issued in May 1979 and bearing a 9 1/8 coupon. Its CUSIP

of a certain debt issue.

In addition to the Treasury, the Federal Reserve Bank buys and sells Treasuries on the open market (through dealers, see below), to implement monetary policy by adjusting the money supply in the US economy.

Treasuries are sold through "single-priced" auctions. Every bid must specify both a price[7] and the quantity desired (expressed in face value), should the bid be successful. After all competitive bids (see below) are submitted, they are ordered in decreasing order based on bid prices. The highest bids are accepted, until the amount of available Treasuries is exhausted.[8] The clearing price is established based on the last (i.e. lowest-priced) accepted offer - everybody pays the same price.

A bid is competitive if it will be used in the determination of the auction price, as described above. An investor submitting a non-competitive bid accepts a priori the outcome of the auction, and assumes the obligation of paying the auction price for the amount bid for. Thus, a non-competitive bid is not used in the process of establishing the auction price. In return for accepting the auction's outcome in advance, a non-competitive bidder receives the assurance that his bid will be honored for sure. A competitive bidder has no such assurance. An investor can submit several competitive bids, in various amounts and at various prices. Non-competitive bids are often used by small retail investors.

Treasuries are issued in denominations of $1,000.

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code is 912810CG1. Source: http://www.publicdebt.treas.gov/sec/seccall5.htm.

[7]Actually, Treasury bids are submitted by specifying yields. Yields move in opposition to prices.

[8]We ignore issues related to the allocation of Treasuries if only part of the last acceptable bids can be satisfied.

1.3 Trading in Treasuries

While the Treasury is the actual issuer of government debt, it is the Federal Reserve System that acts as agent for the Treasury, both at issuance, and later, on the secondary market. In turn, the Fed interacts with a small number of primary dealers, which are institutions pre-approved to handle the Fed's government debt transactions. In return for having such a privileged position, primary dealers assume the responsibility of providing liquidity to the Fed, and of helping it implement its monetary policy goals. Primary dealers engage both in outright buy/sell transactions, and also in repurchase (or "repo") transactions. In a repo transaction the Fed[9] sells Treasuries to a primary dealer with an agreement to buy them back after a period of time, typically a small number of days.

A second tier of market participants is that of inter-dealer brokers and dealers. There is a very small number of inter-dealer brokers (under 10). These consolidate bid and ask offers from dealers, anonymize the data, and distribute the information to their clients. Thus, inter-dealer brokers facilitate market making by disseminating information, but by protecting the anonymity of participants, they also prevent information from being gleaned based on the identity of the traders (Sundaresan 2002, pp. 46 and following).

Dealers, in turn, execute trades on behalf of their clients, which can range from small investors to big institutions.

To simplify accounting for the ownership of Treasuries, these exist only in book-entry form, as records in databases. It is also said that Treasuries are "dematerialized." A multi-level record keeping system exists to record ownership. At

[9]Repos are used extensively by market participants. We omit these details, as we are focusing only on the role of the Fed in the Treasury market.

the top level, each Federal Reserve Bank keeps a record of the aggregate holdings of certain depositary institutions in their district. Depositary institutions that do not have accounts with one of the Federal Reserve Banks must maintain an account with a depositary institution that has such an account. Similarly, at every level down in the hierarchy, the holdings of an institution at a lower level are recorded only in aggregate at the immediately higher level in the hierarchy (Sundaresan, p. 54).

It is a fact not generally known outside the financial community that the aggregate daily volume of trades in government debt (and in general, in fixed-income securities) dwarfs the aggregate daily volume of equity markets. The difference is roughly of two orders of magnitude. While equity trading is dominated by a large number of relatively small trades, fixed-income instruments are traded in large tranches, but in a relatively small number of transactions.

Treasuries - rather, obligations to sell Treasuries at a later date - begin to trade roughly two weeks before a certain issue is made available to the public. This "when-issued" trading makes it possible for market players to better forecast the prices of the respective Treasuries, and to adjust their behavior accordingly.

1.3.1 Treasury Prices

Depending on the maturity of the Treasury at hand, markets use different conventions to quote prices. These conventions are unwieldy, and complicate the uniform handling of Treasury price data.

Daycount Conventions

Due to the non-uniformity of calendar subdivisions and due to a desire to simplify certain computations in a pre-computer age, traders have developed a set of conventions to convert the number of days between two events (say, from the current date to the maturity of a given bill) to years. Treasury traders use two daycount conventions: actual/360, and actual/actual. In the actual/360 convention the actual number of days between two events is counted, and it is converted to years by dividing it by 360. The actual/actual method involves using the actual daycount between two moments in time, divided by the actual number of days in the current year (the current year is defined as the period between the current date and the occurrence of the same date in the immediately following year). "Actual" computations are simplest when dates are expressed as number of days from a base date, and the interval is expressed as the difference between two such dates.

A good reference for formulas used in the fixed-income area is (Lynch and Mayle 1986). We are using the same reference in the two subsections below.

Treasury Bills

Treasury bill prices are on a discount basis, as follows:

$$Price = 100(1 - \frac{DR * DSM}{360}) \tag{1.1}$$

where the price is expressed per \$100 face value, DR is the discount rate (as a decimal, as opposed to percentage), and DSM is the actual number of days between settlement and maturity date. As the reader will note, T-Bills use the actual/360 daycount convention.

The discount rate is the simple interest rate that makes the discounted face value of the bond equal to its price. Discount rates must be converted to dollar prices for the purposes of forward rate smoothing.

Treasury Bonds and Notes

Bond and note prices are often expressed implicitly, by using periodically compounded yields. In such a case, the yield represents the periodically compounded interest rate that discounts the future cash flows of the instrument at hand so that the sum of discounted cash flows equals the price of the respective instrument. For the simplified case of equal periods between cash flows[10] the formula for a note or bond's yield is

$$p = \sum_{i=1}^{n} \frac{c_i}{(1+y)^i} \tag{1.2}$$

where p is the bond price, c_i is the cash flow at time $t_i = i\Delta$, Δ is the interval between cash flows (usually six months), and y is the yield.

Our data source (GovPX[11]) provides the so-called "clean" bond and note prices. A "clean" price is a price that does not include accumulated interest for the current period. In the Treasury market interest is assumed to accumulate linearly between two coupon payment dates. With the usual conventions, a Treasury with a coupon of C pays $\frac{C}{2}$ dollars per each \$100 face value every six months. If PD is the date of the previous coupon payment, ND is the date of the next coupon payment, and CD is the current date, then the accumulated interest from PD to CD is given by $\frac{C}{2} * \frac{CD-PD}{ND-PD}$, where $CD - PD$ and $ND - PD$ represent daycounts between the respective events. The "actual/actual" convention is used.

For our purposes, we compute the full (or "dirty") price of a bond by adding the

[10]This assumption implies that we are just after a coupon has been paid, or on the dated date of the note or bond.

[11]Databases of Treasury information will be discussed in a subsequent section.

clean price and the accumulated interest. This is the only economically meaningful price, and it represents the amount of cash that changes hands in a transaction. The clean price is meaningless beyond its use in this market convention.

1.3.2 Idiosyncratic Factors that Affect Treasury Prices

Certain Treasury issues have characteristics that make them attractive to certain classes of investors. The collective actions of these investors can distort the price of these Treasuries when compared to instruments of similar maturity. We have already mentioned the excess demand for long term STRIPS that makes them relatively more expensive.

At the short end, it has been shown that bills maturing at the end of the month tend to be more expensive than bills maturing earlier in the month. One possible interpretation for this effect is that businesses tend to incur a significant portion of their expenses at the end of the month, therefore they will invest preferentially in instruments that will mature just before these obligations are due (Cook and LaRoche 1993).

It is a consistent characteristic of Treasury markets that the latest issue of a certain maturity trades at a lower yield (hence it is relatively more expensive) than instruments of nearby maturity. These Treasuries are said to be "on-the-run." Treasuries characterized by low yields compared to similar issues are in high demand, it is easy to sell them, and relatively expensive to buy them. Such instruments are said to be "liquid," or to have "high liquidity." Conversely, instruments characterized by excess supply are relatively cheap (have a high yield). Such instruments are said to have "low liquidity." Treasuries that have a long leftover term to maturity, but are far from being on-the-run, tend to be characterized by low liquidity. Also, bonds and notes close to maturity tend to be illiquid.

1.4 Sources of Treasury Data

We will use three data sources of government bond information. These are the University of Houston's Fixed Income Database, the Center for Research in Security Prices (CRSP), and GovPX. We only had access to historical bond prices. We have accessed CRSP and GovPX through the Wharton Research Data Services (WRDS) at the University of Pennsylvania. These databases differ widely in their characteristics both in terms of usability, and in terms of information that can be gleaned from them.

In the case of the University of Houston's database, the data consists of monthly bid prices for various fixed income securities, including U.S. Treasuries and U.S. corporate debt. The bid prices are obtained from Lehman Brothers trading sheets on the last calendar day in each month. For each security included, various identifying information is also provided including embedded options, seniority status, and whether the bid price is transaction based or matrix priced. Additional details are provided in (Warga 2004).

The CRSP database includes daily and monthly Treasury bid and ask quotes, delivery dates, information on other interest rates (1-, 3-, and 6-month certificates of deposit, 30-, 60-, and 90-day commercial paper, federal funds rate). The database also reports information that can be recomputed from fundamentals, such as accumulated interest, yield, return on the bond, and duration. Daily Treasury data begins on June 14, 1961. Monthly data begins on December 31, 1925. CRSP claims to be virtually complete with respect to information on US government debt.

GovPX is, in principle, the most complete database. Because no documentation on GovPX was directly accessible to us,[12] we have collected information from a number of eclectic sources, including insights from the careful analysis of data informed by our general knowledge in the area, fragments of documentation gleaned from the Internet, and other academic papers and books that have reported on GovPX. We base the bulk of the following exposition on (Fleming 2003) and (Boni and Leach 2002).

GovPX was established with the purpose of increasing Treasury market transparency, as a direct response to political pressure brought on by Congress and the SEC. In 1997 all but one inter-dealer broker[13] provided data to GovPX. Treasury trading was traditionally conducted over the phone (these were the "voice brokered operations"). In recent years, however, a significant portion of trading migrated to computerized platforms, and the volume of voice brokered transactions decreased accordingly. By the end of 2000, GovPX lost all access to electronic trade data, and was relegated to report data obtained from voice feeds only. In 2000, all but one inter-dealer brokers provided voice feed data to GovPX.[14] This evolution led to a marked decrease in GovPX market coverage (see below).

GovPX consolidates Treasury trading activity and distributes the data live all over the world. GovPX transaction[15] records contain the best bid and offer

[12]According to a personal communication from Michael Boldin, WRDS' research and support services director, such documentation has not been made available to WRDS despite WRDS' explicit requests. At the time of this note's writing (June 2, 2004) the situation has not been remedied. We quote this to illustrate how difficult it is to get a data vendor's help in the area of financial databases. We have anecdotal evidence, and actual knowledge of similar unresponsiveness by other data providers as well.

[13]The exception was Cantor-Fitzgerald (Boni and Leach 2002).

[14]Again, the exception is Cantor-Fitzgerald (Boni and Leach 2002).

[15]We refer to a transaction in the sense of an event that modifies market information, not in the sense of an actual trade. There are many more transactions

quotes and the corresponding bid and offer quantities, the price and size of the last trade, whether the last transaction was seller or buyer initiated (a "hit", or "take," respectively), a timestamp of each transaction, and other information.

Besides the outcome of a trade, GovPX also reflects the negotiations that occur just before a trade occurs. Traders in this market are concerned that if they place a large order, they will not have time to retract it, should market conditions suddenly change. In such cases a counterparty could quickly match the-by-now stale order, and force the original trader to participate in a losing transaction ("free-option"). Also, traders are concerned that if they place a large order, the size of the order will permit other market players to infer information about changing market conditions, and quickly place a smaller order priced so that it would be executed before the first broker's large order ("free-riding"). To avoid these issues traders prefer to place small, so-called expandable orders, which they can directly negotiate with the counterparty just before they conclude a trade. This process, called "workup," is also reflected in GovPX data.

There is strong evidence that GovPX coverage, while probably the best available commercially, is limited, and rapidly falling. Comparing the aggregate amounts reported by inter-dealer brokers to the New York Fed with transactions recorded by GovPX, Flemming concludes that GovPX market coverage was about[16] 42% in early 2000. The level of market coverage decreased steeply in the recent past, from 65% in 1997, to 57% and 52% in 1998 and 1999, respectively. In this work

than trades.

[16]Flemming points out a few methodological limitations of his estimate, primarily due to the fact that inter-dealer brokers only report aggregate trade volume to the Fed. Thus, a typical transaction would be reported twice, by both participating inter-dealer brokers, resulting in a double count of the aggregate transaction volume. However, a small number of transactions is executed with non-inter-dealer brokers, and these, in turn, are not double counted at the Fed level.

we are studying the US Treasury market during 1999 and 2000, so these estimated coverage levels are directly relevant to us. We are not aware of more recent market coverage estimates.

Further, there is evidence (Boni and Leach 2002), also mentioned by (Fleming 2003), that the lack of coverage is biased with respect to the Treasury's maturity. It is estimated that market coverage of Treasury bill transactions was over 90% in all years from 1997 to 2000. As opposed to this, the estimated market coverage of coupon securities with 5 years or less to maturity fell from 70% in 1997 to 39% in early 2000. For coupon securities over 5 years or less to maturity fell from 37% to 19% in the same interval.

The relatively small - and decreasing - market coverage of GovPX is a source of concern, especially at the long end. The impact on the results, however, depends on the specific problem and methodology that is employed in a particular study.

In our study, we are only considering the last actual trading price for each bond for each calendar day. Because bond trades do not occur simultaneously, and certain bond issues do not trade very frequently, our approach introduces a source of error due to the non-contemporaneous price data that we use.[17] On the other hand, most trades involve on-the-run Treasuries, both at the short, and at the long end. Market coverage is not as much of a concern as it might seem, principally because most of the trades are executed on on-the-run Treasuries. If GovPX reports a bond price that corresponds to a trade late in the day, and prices have not moved significantly, it does not really matter that we might have missed most of the transactions on the respective bond.

Historical financial databases are notoriously unreliable, due primarily to hu-

[17]This approach is implicitly accepted by researchers working in bond smoothing, so much so that they do not even note the potential problem.

man error. Such errors can occur when data is entered into an electronic trading system by the traders themselves, but errors can be introduced at later stages as well. Sometimes trading occurs by voice, where transcription errors are frequent. We have anecdotal evidence that indicates multiple transcription stages, e.g. situations when an individual writes down prices from a proprietary data provider's datafeed, enters the data again, by hand, into another system, which then further disseminates the data. Issues like this occur because of technical incompatibility between various datafeeds, or for copyright reasons.

From a data vendor or trader's perspective data is most valuable during a short time frame immediately after it has been generated. Once the data becomes stale with respect to the market, the vendor's motivation to find and fix the errors is very low. It is the users of historical financial data who face the burden of detecting and coping with these problems. If errors affect critical information that is otherwise unrecoverable from the data, the researcher is faced with the always difficult problem of convincing a data vendor to correct the error.

It is common to have historical financial data stored and distributed in the form of plain text files. This minimizes platform incompatibility issues, but makes data corruption very easy. As technologies and underlying assumptions change, errors can be introduced in subtle and sometimes hard to detect ways. If, for example, a space in a particular part of the input is interpreted as a space by one program that processes the data, but as a 0 by another program, possibly introduced years later and based on a very different technology, the original connection between two parts of the database will be lost.[18] Such problems can be very hard to detect unless the data user actually anticipates and tests for them. Some errors can be subtle and

[18] For example, a bond might "split" into two, or might "disappear" and "reappear" later.

expensive to detect - and complicate to code - which often makes programmers ignore them. In the recent past this author has helped a fellow programmer detect 30-40 types of errors in an established financial database within a matter of weeks. The data vendors we are familiar with do not maintain a documentation of errors they have corrected, and neither do they maintain a library of error types detected. The latter would be of great pragmatic use for researchers examining historical financial data.

1.5 Characterization of Data Sets used for Treasury Smoothing

We will examine first the University of Houston's Fixed Income Database, covering the period between May 1991 and March 1997. Within this period, we have eliminated all Treasuries with unusual features (e.g. callable bonds, or flower bonds). We have also eliminated Treasuries with matrix prices[19], and Treasuries with prices so high, or so low, that were clearly incorrect. In general, we have eliminated all records that had at least one suspicious or incorrect field, even if that field was not necessarily a price field. Using a median yield filter of 2.5% we have eliminated instruments whose yield differed by more than this limit from the median yield of their immediate neighbors.[20] This filtering is only likely to detect the most egregious errors. In general, such errors occur with instruments with very short maturities left. A small price error here can change the yield significantly.

[19] These are prices that are inferred from the prices of nearby, "similar" instruments. Not being actual prices, we consider matrix quotes unreliable, as likely to convey the biases of the model used to generate them, as they are likely to provide useful price information.

[20] We used the five neighbors with immediately shorter maturity, and five neighbors with immediately larger maturity.

Table 1.1: Annual averages of the number of Treasuries used for smoothing, and their distribution in terms of leftover term to maturity, after filtering. All numbers are rounded to the nearest integer. Source: University of Houston Fixed Income Database.

	Total	**Leftover Maturities in Years**						
		(**0-.5**]	(**.5-1**]	(**1-3**]	(**3-5**]	(**5-10**]	(**10-20**]	(**20-30**]
1991	190	19	17	57	36	30	14	17
1992	222	42	24	56	37	32	11	20
1993	224	42	24	56	38	34	8	22
1994	198	23	19	56	40	32	5	23
1995	212	35	22	59	39	31	3	23
1996	217	42	24	59	38	28	5	21
1997	213	36	25	60	37	27	6	22

At the long end of the curve, this is much less of a problem. After filtering, we have retained approximately 29,100 Treasury price records, which we then used for smoothing. We have ignored minor inconsistencies in the data.[21] A summary of the available number of bonds in the database is provided in table 1.1.

We have used the University of Chicago's Center for Research in Security Prices (CRSP) as the source of individual Treasury characteristics (issue date, maturity date, coupon, etc.). We have also used CRSP to obtain daily bid and ask quotes for all outstanding Treasuries with no special features. The time period we have

[21]For example, each monthly dataset is assigned to the last calendar day of the respective month, even if that day was not a business day. This created difficulties when matching Treasury data with stock market data in our credit risk projects. An instance of this problem occurred on November 30, 1996, which was a Saturday.

Table 1.2: Annual averages of the number of Treasuries used for smoothing, and their distribution in terms of leftover term to maturity, after filtering. All numbers are rounded to the nearest integer. Source: CRSP.

	Total	**Leftover Maturities in Years**						
		(**0-.5**]	(**.5-1**]	(**1-3**]	(**3-5**]	(**5-10**]	(**10-20**]	(**20-30**]
1999	207	44	25	53	28	22	11	24
2000	184	43	21	45	21	17	14	23

included in our smoothing study covers the full years of 1999 and 2000. As the reader will note, this period does not overlap with the corresponding period of the University of Houston's database. This is because we have no access to more recent University of Houston data. We summarize the distribution of used bonds in table 1.2. The number of Treasuries available in the database is changing very slowly within the years under study, so that over the long term there is a slight decrease in the number of outstanding bonds. This probably reflects the improved fiscal equilibrium of the US Government toward the end of the 1990's.

GovPX, by comparison reports a very small number of trades for bonds at the long end. The number of price quotes, after eliminating the transactions that are likely to contain errors is summarized in table 1.3.

We have quoted above a reference that estimates GovPX coverage to have been approximately 65% overall and in the range of 40% at the long end in the year 1999. At first sight the difference between these percentages and tables 1.2 and 1.3 is striking. To understand what is happening, we point out that the CRSP database contains quotes for all outstanding Treasuries, whether they actually traded or

Table 1.3: Annual averages of the number of Treasuries used for smoothing, and their distribution in terms of leftover term to maturity, after filtering. All numbers are rounded to the nearest integer. The reader should note the marked decrease in the total number of available bonds from 1999 to 2000. Source: GovPX.

	Total	**Leftover Maturities in Years**						
		(**0-.5**]	(**.5-1**]	(**1-3**]	(**3-5**]	(**5-10**]	(**10-20**]	(**20-30**]
1999	92	34	18	36	2	1	0	1
2000	67	31	14	19	1	1	0	1

not on a given day. In addition, the coverages as reported by (Fleming 2003) and (Boni and Leach 2002) refer to GovPX coverage in terms of total volume, not number of actual trade prices quoted. Because most trading occurs with on-the-run Treasuries, a large part of the traded volume is masked by the existence of only a few on-the-run prices listed in table 1.3. Thus, our tables do not contradict an average 40% coverage of trades at the long end. We do not have a means to independently check for this coverage ratio, however, as we do not have access to the aggregate trading information that the inter-dealer brokers report to the Fed.

Table 1.4 analyzes in more detail the characteristics of the bond collection on a typical (business) week in our sample (June 21-25, 1999). A Treasury is "good" if it does not have special features (e.g. it is not callable), has non-contradictory trading history, and at least one trade was executed on it on the given day. A bond is "acceptable" if it is not callable, has a non-contradictory trading history, has bid and/or ask transactions, but no actual trade was executed on it. A bond is "bad" if it has special features (typically: callability), if none of its transaction records

Table 1.4: Number of good, acceptable, and bad Treasuries, as reflected by GovPX.

Date	Total	Good	Acceptable	Bad
1999-06-21	208	79	42	87
1999-06-22	208	102	21	85
1999-06-23	208	99	26	83
1999-06-24	209	102	21	86
1999-06-25	208	92	34	82

contains (at least) some valid bid or ask prices,[22] if all the transaction records in which it appears contain an incorrect field value, or if its trading history appears to be contradictory. An example of contradiction is when total trading volume dips after a transaction, rather than going up.

Almost all good bonds are used in smoothing; but before they are inserted into the bond collections for a particular day, they are further carefully tested for consistency of their parameters.[23] Bonds are eliminated in this stage, for example, if their dated date is on or after their maturity date, or if their price is so much out of the expected range that they are clearly incorrect. Typically, at most one or two bonds per day are eliminated using these tests.

[22]There exist bonds with many associated transactions that do not contain actual price information. An example would be the bond with CUSIP 918210CZ9 on June 22, 1999. This bond has 163 records associated with it, but no transactions, and no actual bid/ask prices. The records contain, however, "indicative ask" and "indicative bid" entries. These appear to be "suggested" valuations provided by GovPX based on their proprietary valuation models. As they do not represent actual prices, and they might express the biases of a particular model, we do not consider them.

[23]Recall that the previous tests referred to the consistency of trading; here we look at the bond itself.

Table 1.5: Distribution of leftover maturities for bad and acceptable bonds on June 22, 1999. Source: GovPX.

	Total	**Leftover Maturities in Years**						
		(0-.5]	**(.5-1]**	**(1-3]**	**(3-5]**	**(5-10]**	**(10-20]**	**(20-30]**
Bad	85	2	0	4	26	19	11	23
Acceptable	21	5	5	9	0	2	0	0

We will now analyze the maturity distribution of the 85 bad, and that of the 21 acceptable Treasuries of June 22, 1999, shown in table 1.5.

As far as we know there are no better sources of actual historic Treasury price data that are accessible to academic research. The lack of bond pricing data at the long end is insurmountable at this stage, so we will have to explore methods that accommodate it, or we will have to use quoted prices.[24]

To complete our study on sources of Treasury price databases, we will now provide a number of graphs that capture a few key characteristics of the Treasury market, as reflected in GovPX and CRSP. The graphs illustrate the cross-sectional distribution of bid/ask spreads and maturities, the typical relationships between CRSP and GovPX price data, the intra-day trading history of a Treasury as reflected by GovPX, the intra-day evolution of the aggregate trading volume[25] of a heavily traded bond, and the evolution across time of a bill's price and bid/ask

[24]We believe quotes should not be used, at least not directly. We will return to this point later, when discussing the practical computation of smoothest forward rate curves.

[25]GovPX has ceased reporting the aggregate trading volume of Treasuries as of 2002 (personal communication by David A. Robinson at WRDS, relaying information he obtained from GovPX following up on my requests for clarification).

spread. All the characteristics illustrated here are typical (within their own categories), and represent well the period 1999-2000.

Figure 1.1 illustrates the relationship between Treasury prices as reflected in CRSP and GovPX. Bonds are represented in the order of their leftover maturity, but the horizontal axis is not proportional. This representation artificially stretches out the left end of the curve, but it allows for a better grasp of the price relationships that we are studying. Note that the CRSP bid/ask prices are so close that they are virtually indistinguishable at the scale used in this graph. For historical reasons, bonds and notes are quoted in dollar 32nds, while bills are quoted in basis points. Figure 1.2 represents the same information, but it uses a linear (thus, proportional) horizontal axis. The reader should note how skewed the distribution of leftover bond maturities is toward the left end (short maturities). Big gaps exist in the CRSP data between approximately 10 and 17 years. This gap is due to callability features embedded into bonds with leftover maturities in this range. Table 1.6 shows the callable bonds recorded as outstanding on June 22, 1999 by CRSP. Callable bonds are not used in our study. To use them, we would have to provide a pricing model for callable Treasuries; we do not attempt that here. The gaps are especially striking in the GovPX data, where practically only the on-the-run bonds with maturities of 5, 10, and 30 years are available.

Figure 1.3 shows the distribution of leftover maturities and prices on a non-proportional horizontal axis (being similar in this respect to figure 1.1). This figure also shows the distribution of bid/ask spreads. The graph clearly shows that bid/ask prices for bonds are quoted in dollar 32nds; thus, these spreads are relatively large. Bid/ask prices for bills are quoted in basis points, and thus are, by comparison to bonds, very small.

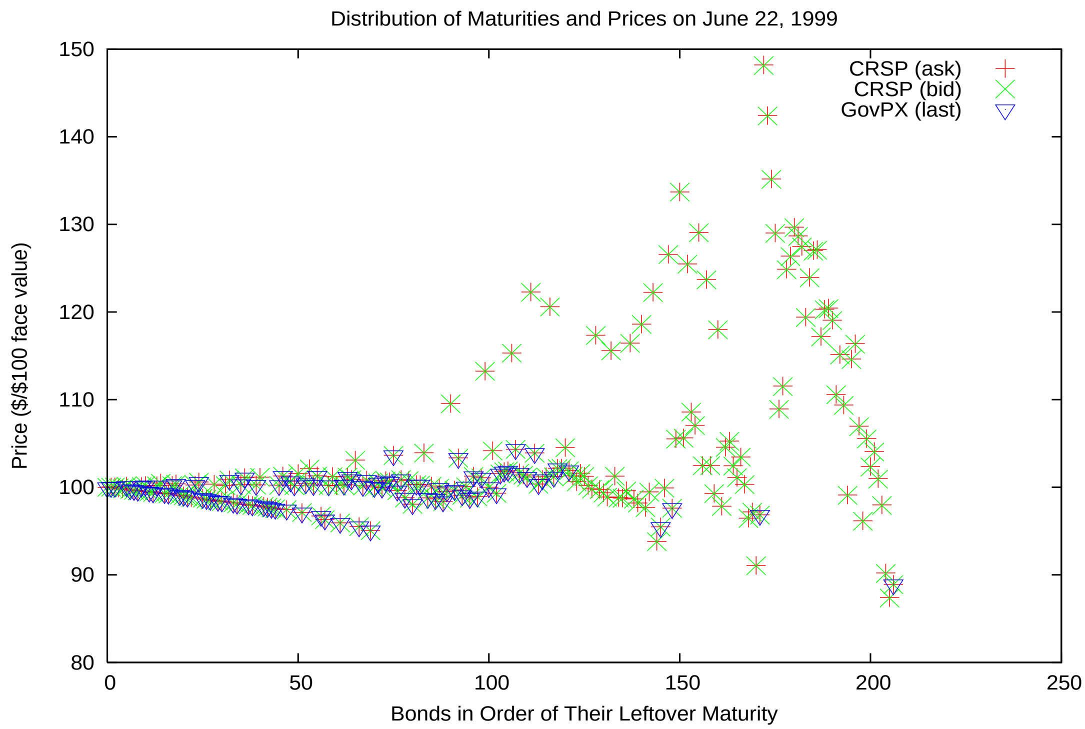

Figure 1.1: This graph illustrates the relationship between Treasury prices as reflected in CRSP and GovPX. The left end is artificially stretched out using a non-linear horizontal scale, where bonds are ordered by leftover time to maturity. At this price scale, the bid/ask spread is almost indistinguishable (bond and note prices are quoted in dollar 32nds). Prices are shown ex-accumulated interest.

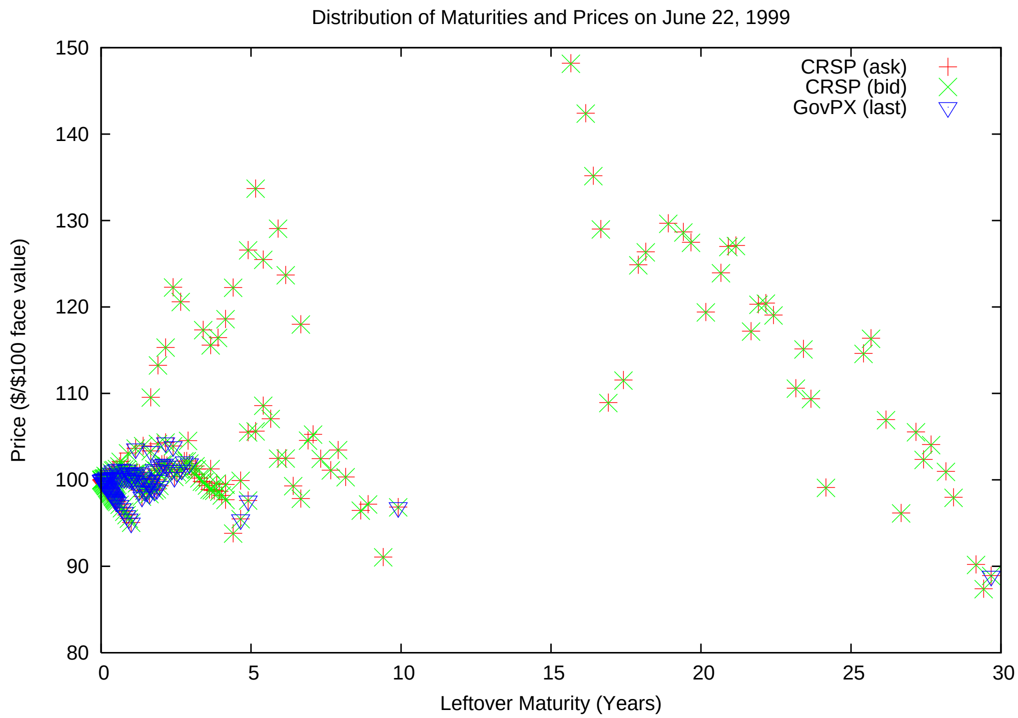

Figure 1.2: This graph illustrates the relationship between Treasury prices as reflected by CRSP and GovPX. Leftover time to maturity is shown on a proportional horizontal scale. Note the large gaps in available GovPX price information. Prices are shown ex-accumulated interest.

Table 1.6: Outstanding callable bonds, as reported by CRSP, on June 22, 1999. The majority of these bonds have leftover maturities between 10 and 15 years, corresponding to the gap in the CRSP price data. There are no bonds without special features in the gap. The bond with CUSIP 9128275G, with maturity on 05/15/2009 is the last regular bond before the gap; the bond with CUSIP 912810DP, with maturity on 02/15/2015, is the first after the gap.

CUSIP	**Maturity Date**	**CUSIP**	**Maturity Date**
912810BS	02/15/2000	912810CM	02/15/2010
912810BV	08/15/2000	912810CP	05/15/2010
912810BW	08/15/2001	912810CS	11/15/2010
912810BU	05/15/2005	912810CV	05/15/2011
912810BX	02/15/2007	912810CY	11/15/2011
912810BZ	11/15/2007	912810DB	11/15/2012
912810CC	08/15/2008	912810DF	08/15/2013
912810CE	11/15/2008	912810DJ	05/15/2014
912810CG	05/15/2009	912810DL	08/15/2014
912810CK	11/15/2009	912810DN	11/15/2014

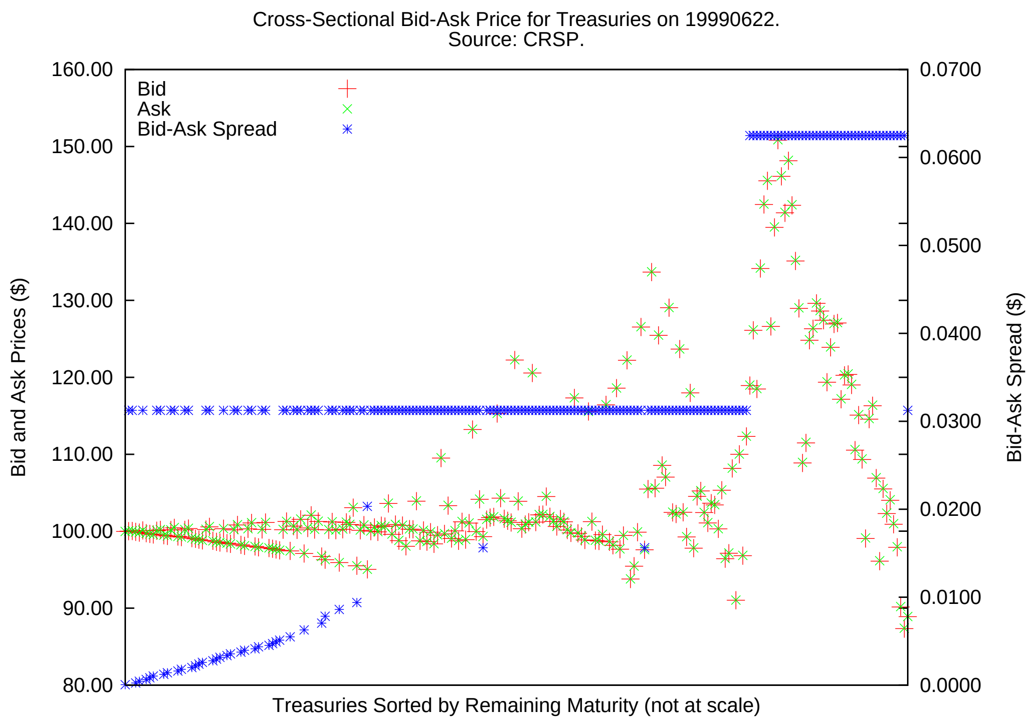

Figure 1.3: A representation of bid/ask price relationships as reflected in CRSP. The left end of the curve is stretched out by not using a proportional horizontal axis. The bid/ask spreads for notes and bonds are expressed in dollar 32nds (only two multiples of this value occur here). Spreads for bills are expressed in basis points, and are typically much smaller than in the case of bonds.

We study the consistency of prices in CRSP versus GovPX in figure 1.4. Ideally, the double inequality $p_{CRSP\ bid} \leq p_{GovPX} \leq p_{CRSP\ ask}$ should hold for every instrument, and all represented spreads should be strictly non-negative. At the short end of the curve, however, the CRSP bid price is higher than the GovPX price. This does not prove that the data is invalid; it is probable that most of the differences arise because the CRSP quotes are not contemporaneous with the GovPX data. As the reader will recall, we use the last trade of the day for each Treasury. If, as it it typical, liquidity decreases slightly toward the end of the day, prices of actual transactions will go down, and might dip slightly under prices prevalent earlier in the day. Figure 1.4 likely captures just this phenomenon.

Figure 1.5 shows the typical trading history of a lightly traded bond, as reflected in GovPX. Most significant events in the trading history are, as expected, bid/ask quotes. The heavy dotted line illustrates the evolution of the bid/ask spread for those GovPX records that contain both bid and ask quotes. If one of the bid or ask prices is not available, the last seen ask or bid price, respectively, is extrapolated to produce a bid/ask spread. These extrapolated spreads are shown by the light dotted line. A thick line is used to indicate the evolution of this instrument's intra-day total trading volume. Only five trades are recorded for this bond on this day, but the aggregate transaction volume is about $45 million.

There are Treasuries that are traded much more heavily than the instrument whose trading history is represented in figure 1.5. For the purposes of our exposition, however, it is not practical to provide an analogous graph showing a complicated trading history; it would be illegible. We restrict ourselves to show the aggregate trading volume for such a heavily traded Treasury in figure 1.6. The differences versus the lightly traded bond are striking - GovPX records almost

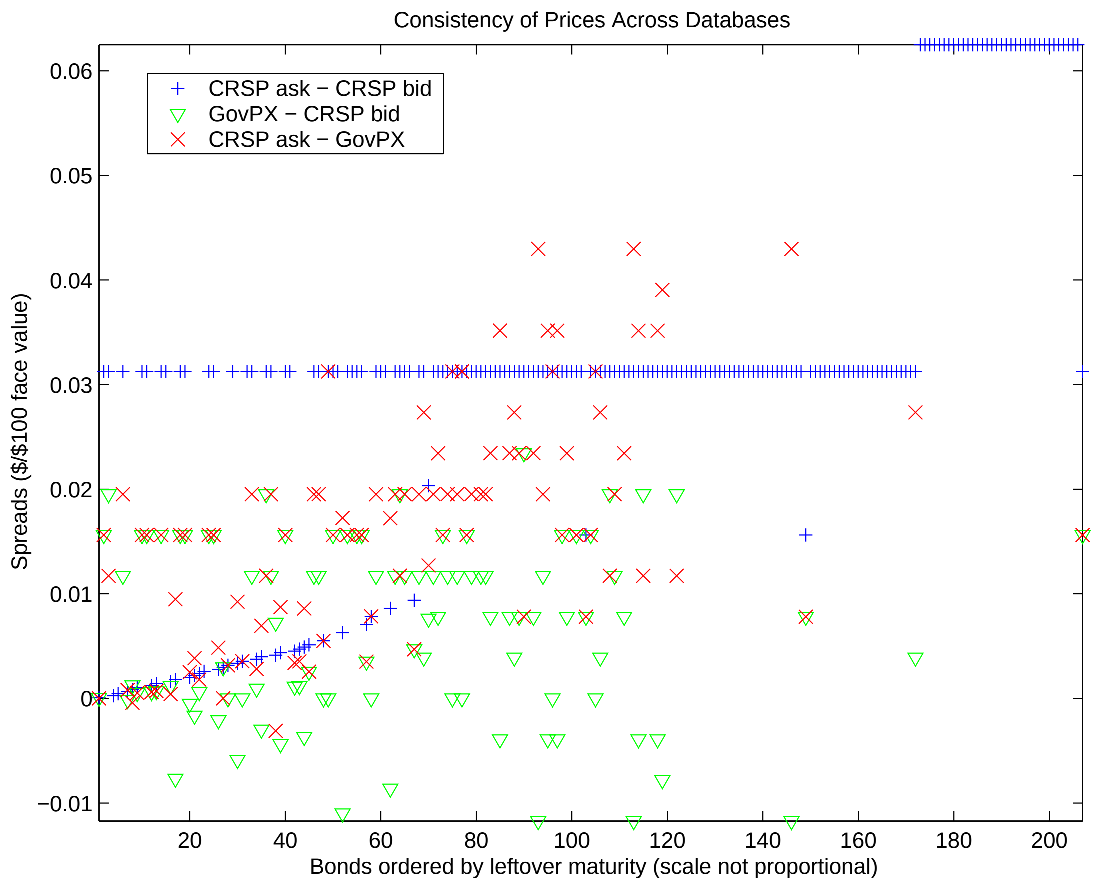

Figure 1.4: The figure illustrates Treasury price consistency problems that arise between CRSP and GovPX. All represented spreads should be strictly non-negative, however, on the short end the GovPX price occasionally dips under the CRSP bid price. Prices are shown ex-accumulated interest.

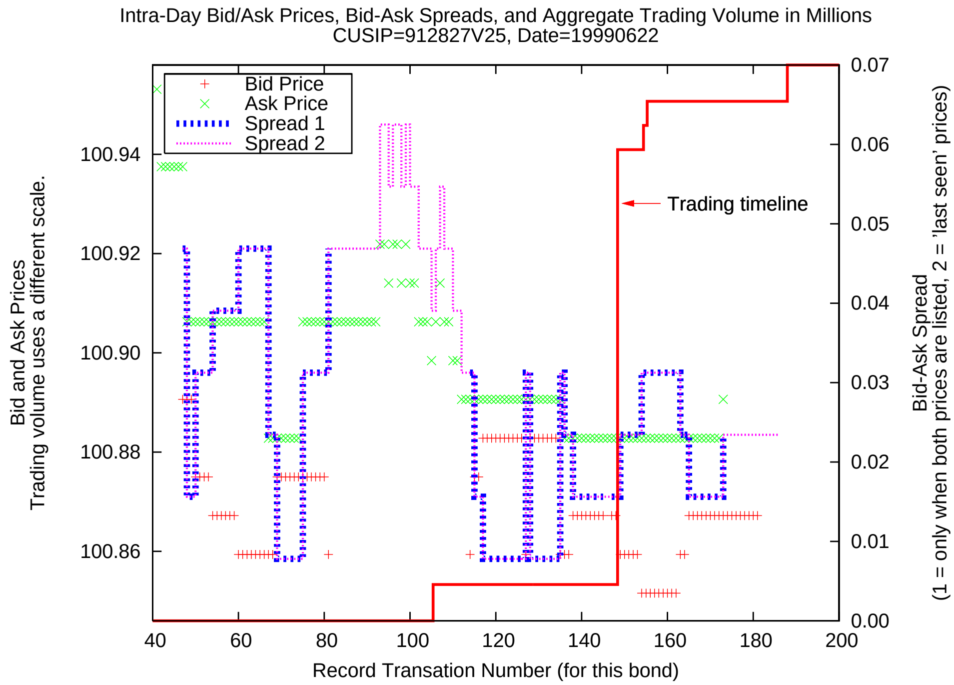

Figure 1.5: The trading history of a lightly traded bond as reflected in GovPX transaction records. "Trading volume" represents the intra-day aggregated trading volume. "Spread 1" represents the evolution of the bid/ask spread over time. When transaction records do not provide both bid and ask prices, the missing value is extrapolated by using the last seen bid/ask price, whichever is needed. This extrapolated line is shown by "spread 2." At the end of the day, the aggregated trading volume for this instrument was just under $45 million.

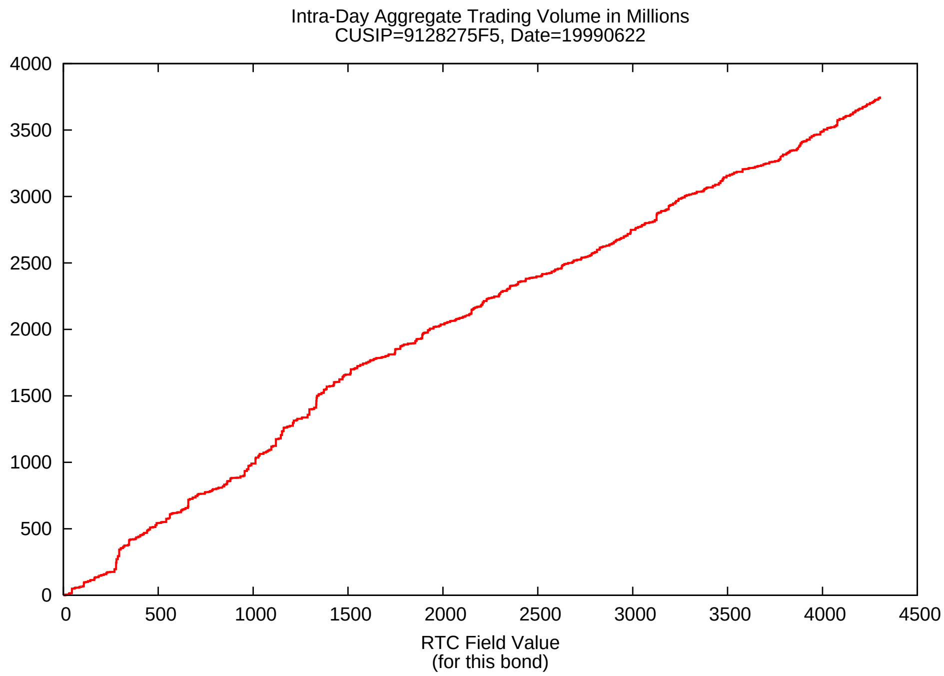

Figure 1.6: The aggregated trading volume, in face value, for a heavily traded Treasury. Note that the total volume reaches almost $4 billion. There are almost 4,500 transaction records for this bond, but not all of them are trades (e.g. some are bid/ask quotes). The 123 acceptable bonds on this day have an average of 368 transaction records each, with the minimum equal to 46 transactions, and the maximum equal to 4311 transactions. Each bond transaction is indexed by a "record transaction number" (RTC), which helps reconstitute the history of the respective bond. It is not rare that a few missing RTCs are inserted into the database toward the end of the trading day, to complete the trading history.

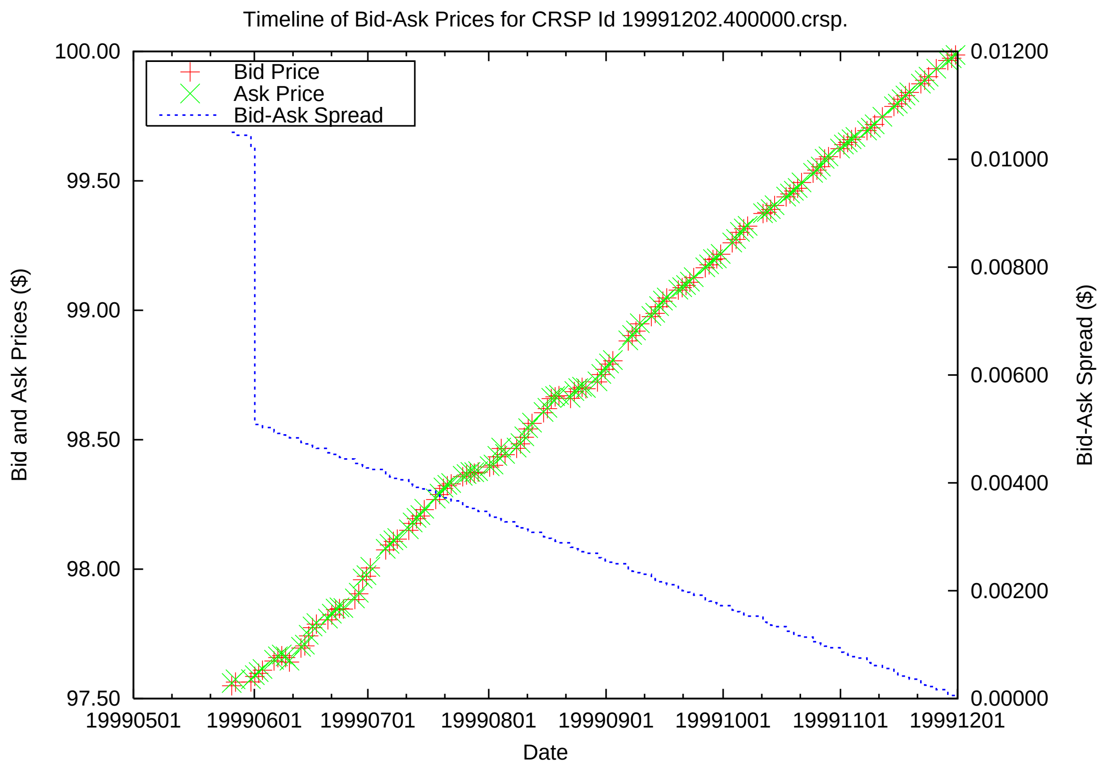

Figure 1.7: Evolution of CRSP bid/ask price history for a 6-month Treasury bill. Note the almost linear increase in price over this short interval, and the convergence of the price at par. The spread converges to 0 almost linearly. We have no good explanation for the extremely regular evolution of the bid/ask spread. We suspect that such a long term predictable pattern is an artifact. All bill prices reported by CRSP have this feature. The first part of the trading curve occurs before the bill starts to accumulate interest. The bill is dated on June 3, 1999, and accumulates interest after this date, but it has been quoted since May 27, 1999.

4,500 transactions for this bond, with an aggregate trading volume of almost $4 billion. For the 123 "good" and "acceptable"[26] bonds on this day, the database records an average of 346 transactions per bond. On this day, the smallest number of record transaction for the 123 bonds considered was 368, while the largest such number was 4,311.

In the interest of brevity we will forgo further detailed analysis of the dataset. Our brief survey revealed the practical difficulties of acquiring, appropriately filtering, and using historical Treasury price information. Using GovPX one can, in principle, acquire a full view of an instrument's trading history, including the workup process just before a trade is executed. GovPX, however, suffers from relatively low coverage, and contains many inconsistencies. CRSP, on the other hand, is complete, but its prices are mostly quotes, which are likely not to always reflect the true market value of a security. The University of Houston's database shares many of its characteristics with CRSP, but it has the further drawback of listing only monthly data.

[26]We remind the reader that an "acceptable" bond is one for which we have no actual trade record, but for which we have bid/ask quotes, and whose trading history is not otherwise inconsistent.

Chapter 2

Forward Rates

2.1 Definitions and Basic Properties of Forward Rates

We defined a riskless (or default-free) zero-coupon bond to be an instrument that pays one sure dollar at a given future moment in time T. The time-t price of this instrument will be denoted by $p(t, T)$.

As defined here, $p(t, T)$ expresses the time value of money for default-free borrowing in the interval $[t, T]$. We will also call $p(t, T)$ the discount factor corresponding to sure cash flows at times T. If there are no arbitrage opportunities in the economy, then $p(t, t) > 0$. Also, we must have that $t_1 < t_2 \Rightarrow p(t, t_1) > p(t, t_2)$.[1]

The term structure of bond prices at time t is given by function $p(t, x)$, where $x \in [t, T_{\max}]$. The time horizon $T_{\max} - t$ is not well defined; in empirical studies it is usually assumed to be in the range of 30 years. This function can not be directly observed in the market, except - maybe - in the case of stripped bonds. There is evidence, however, that the price of stripped bonds is distorted by excess demand, so their prices might not be good approximations of the corresponding discount factors.

The forward rate $f(t, T)$ is defined by

$$f(t, T) = -\frac{\partial \log p(t, T)}{\partial T} \tag{2.1}$$

Finance theory does not impose too many constraints on f. It is generally

[1] If one can borrow at the riskless rate, and $p(t, t_1) < p(t, t_2)$, then one can buy a bond paying \$1 at time t_1, and sell a bond that pays \$1 at time t_2. This generates an immediate sure profit of $p(t, t_2) - p(t, t_1)$. At t_1 the proceeds from the maturing bond are cashed and held until they are used to pay the outlay on the bond maturing at t_2.

accepted that f is continuous (Heath, Jarrow, and Morton 1992), and that it is "smooth." In practice, f is assumed to have all the continuity and differentiability properties that are necessary for its use in a particular context.

We immediately have that

$$p(t,T) = \exp(-\int_t^T f(t,\tau)d\tau) \tag{2.2}$$

The short (or spot) rate at time t is defined as $r(t) = f(t,t)$.

Forward rates can not be directly observed in the market. Arbitrage arguments in an economy that contains cash currency imply that f must be a non-negative function.[2]

A default-free coupon-bearing bond[3] B with maturity t_B is a package of $n \geqslant 2$ riskless cash payments (c_i, t_i), where positive cash flow c_i occurs at time t_i ($t_{i-1} < t_i$, for all $i = 2, ..., n$; $t_n = t_B$). If we assume that no arbitrage opportunities exist,[4] then the price of the bond is given by:

$$p_B(t) = \sum_{i=1}^{n} c_i p(t,t_i) \tag{2.3}$$

Given the schedule of payments for a bond, formula 2.2 immediately establishes a relationship between bond prices and forward rates.

$$p_B(t) = \sum_{i=1}^{n} c_i \exp(-\int_t^{t_i} f(t,\tau)d\tau) \tag{2.4}$$

---

[2]Assuming that f is negative at time τ, then it will be negative on an open interval (τ_a, τ_b). This, in turn, implies that $p(t,\tau_a) < p(t,\tau_b)$. An individual at time t can lock in a profit by buying a \$1 bond maturing at τ_a and selling a \$1 bond maturing at τ_b. This individual realizes a profit of $p(t,\tau_b) - p(t,\tau_a)$ at time $t = 0$. At time τ_a the individual cashes in the \$1 from the bond maturing at that time, holds on to this amount until τ_b, then uses it to pay off the bond maturing at τ_b. This individual has just exploited an arbitrage opportunity.

[3]We will usually not state that a bond is "coupon bearing." The reader should assume that all bonds are coupon bearing, unless explicitly stated otherwise.

[4]See below for a formal definition and discussion of arbitrage.

Treasury bond prices can be observed in the market.We will denote the observed market price of a bond B at time t by using the symbol $p_B^m(t)$. As shown when discussing the sources of historical Treasury data, the distribution of observed Treasury prices is highly skewed: most traded instruments have short (leftover) maturities, while only a few bonds with (leftover) maturities over 5 years trade on any day. Filling the gaps with Treasury price quotes is problematic, due to the complex relationships that exist between quoted prices and actual prices.

Forward rates are non-constant and hard to describe synthetically. To alleviate this, market players often prefer to use the simplifying assumption that forward rates are constant. The (continuously compounded) yield $y_B(t)$ of bond B at time t is defined as the (assumed) constant forward rate that makes the theoretical price of the respective bond equal to its observed market price:

$$p_B^m(t) = \sum_{i=1}^{n} c_i \exp(-(t_i - t)y_B(t)) \tag{2.5}$$

As yields are defined by an implicit equation, an appropriate numerical procedure is needed to compute $y_B(t)$. Given the simplicity of the functions involved, this does not pose a significant problem in practice. Traders often use yields that are not based on continuously compounded interest. These alternatives provide no additional useful insights; we will ignore them for the rest of our exposition.

In general, the yield at time t of a coupon bond with maturity t_B does not coincide with the yield of a zero coupon bond with the same maturity. The yield of the latter instrument can be written down explicitly as

$$y_0^{t_B}(t) = \frac{1}{t_B} \int_t^{t_B} f(t, \tau) d\tau \tag{2.6}$$

Depending on the characteristics of the forward rate curve, the relative magnitude $y_0^{t_B}(t)$ and $y_B(t)$ can be arbitrary. In the simplifying case of a strictly upward sloping forward rate curve, assuming that the market price of the bond is the same as the price computed based on 2.4 (i.e. assuming that $p_B^m(t) = p_B(t)$), we have the relation $y_0^{t_B}(t) > y_B(t)$. Thus, if one were to represent on the same graph both the forward rate curve $f(t,T)$ and the yields of various coupon bonds, then the yields of the coupon bonds would lie under the forward rate curve. Let us assume for a moment that the yield of the coupon bond $y_B(t)$ is the same as the yield of a zero-coupon bond of the same maturity $y_0^{t_B}(t)$. The price $p_B^*(t)$ of the bond under these assumptions is given by the formula below:

$$p_B^*(t) = \sum_{i=1}^{n} c_i \exp(-\frac{t_i - t}{t_B - t}\int_t^{t_B} f(t,\tau)d\tau) = \sum_{i=1}^{n} c_i \exp(-(t_i - t)y_0^{t_B}(t)) \qquad (2.7)$$

Due to our assumption that the forward rate curve is strictly upward sloping, we have that

$$(t_i - t)y_0^{t_B}(t) = \frac{t_i - t}{t_B - t}\int_t^{t_B} f(t,\tau)d\tau > \int_t^{t_i} f(t,\tau)d\tau \qquad (2.8)$$

This can be seen immediately by dividing both sides of the inequality by $t_i - t$; we then only need to compare the average of the forward rate on the interval $[t, t_B]$ and $[t, t_i]$, respectively.

Because all but the last cash flows in formula 2.7 are discounted at a constant rate higher than that required in formula 2.4 for the equality of the computed and observed bond prices to hold, we infer that $p_B^*(t) < p_B^m(t) = p_B(t)$. This means that the discount factors have been too high, which immediately implies that the yield of the coupon bond must be less than the yield of the zero-coupon bond with the same maturity. In practice, an assumed forward rate curve will only price a bond approximately, and the forward rate curves will not necessarily be upward sloping.

2.2 Theories of the Term Structure of Forward Rates

A number of theories have been put forward to explain or predict the shape of the forward rate curve. We will now examine a few of these theories. Assuming that a reasonably accurate theory of the forward rate curves can be developed, its conclusions would inform the development of practical procedures aimed at finding the actual form of the forward rate curve. In discussing these theories we follow (Jarrow and Turnbull 1999) and (Sundaresan 2002), but we modify the arguments and the formulas to accommodate our use of continuously compounded interest rates.

Expectation Hypotheses

It is commonly held that the shape of the forward rate curve predicts the future evolution of the interest rates.

One formulation of the present value form of the expectation hypothesis states that the present value of a zero-coupon bond is equal to the expected amount of money that must be continuously reinvested for infinitesimal periods of time $[\tau, \tau + d\tau]$ at the short rate prevailing at time τ, $f(\tau, \tau)$. The expectation $E_t(\bullet)$ in this formula is taken with respect to the true probability distribution of events in the underlying economy.

$$p(t,T) = E_t[\exp(-\int_t^T f(\tau,\tau)d\tau)] \tag{2.9}$$

This form of the expectation hypothesis implies that bond prices embed information on both the level of future short rates, and on the probability distribution that will influence their evolution.

The unbiased forward rate form of the expectation hypothesis states that $f(t, T)$

is an unbiased estimate of the future short rates. In other words, we must have that

$$f(t,T) = -\frac{\partial \log p(t,T)}{\partial T} = E_t(r(T)) \tag{2.10}$$

where the $E_t(\bullet)$ again represents the expectation operator with respect to the actual probability distribution of events in the economy. Traders commonly hold the belief that the slope of the forward rate curve predicts the future evolution of short rates (e.g. an upward sloping forward rate curve predicts increasing spot rates). This is an immediate consequence of formula 2.10.

Formula 2.10 implies a pricing formula for bonds under these hypothesis:

$$p(t,T) = \exp(-\int_t^T f(t,\tau)d\tau = \exp(-\int_t^T E_t(r(\tau))d\tau \tag{2.11}$$

Because in general $E_t[\exp(-\int_t^T f(\tau,\tau)d\tau)] \neq \exp(-\int_t^T E_t(r(\tau))d\tau$, the two forms of the expectation hypothesis are not equivalent, and they are exclusive.

Empirical studies have disproved the validity of both the present value form, and that of the unbiased forward rate form of the expectation hypothesis.

Liquidity Premium Hypothesis

The liquidity premium hypothesis states that the equality in 2.10 does not hold, but that the equality can be reestablished by writing

$$f(t,T) = E_t(r(T)) + \pi(t,T) \tag{2.12}$$

where $\pi(t,T)$ is a liquidity premium depending only on $T - t$. The liquidity premium expresses the fact that lenders prefer to extend short term loans, while borrowers prefer to get long term loans. Lenders can be induced to make longer-term loans only if they receive an appropriate liquidity premium.

Segmented Market Hypothesis

The segmented market hypothesis assumes that investors have preferred lending and borrowing horizons, and that the shape of each segment of the forward rate curve is determined in relative isolation by the group of investors that trade within the given segment. A connection with the liquidity premium hypothesis can be established by assuming that a liquidity premium can induce lenders to shift from their preferred trading horizon to another one. A major consideration in this area is that related to various tax constituencies, whose time horizon preferences will be strongly shaped by the prevailing legal framework.

Local Expectation Hypothesis

The local expectation hypothesis states that the expected returns of all assets (bonds, in our case) at time t is equal to the short interest rate:

$$\frac{E_t(dP(t,T))}{P(t,T)} = f(t,t)dt \tag{2.13}$$

This relation has been investigated in (Cox, Ingersoll, and Ross 1981), where it is concluded that only this hypothesis is consistent with no arbitrage. Other researchers have contradicted this assertion; most notably Longstaff shows in (Longstaff 2000) that all forms of the expectation hypothesis can hold in an incomplete economy. Completeness is a standard assumption in financial theory, but empirical research has also revealed hints of market imperfections.

2.3 Term Structure Modeling at Selected National Banks

If one examines table 2.1, which summarizes the current practice of forward rate fitting at the national banks of the major industrialized countries, the most obvious

Table 2.1: Forward rate curve procedures used by some industrialized countries. Source: Filipović, Consistency Problems.

Central Bank	**Curve Fitting Procedure**
Belgium	Nelson-Siegel, Svensson
Canada	Svensson
Finland	Nelson-Siegel
France	Nelson-Siegel, Svensson
Germany	Svensson
Italy	Nelson-Siegel
Japan	Smoothing splines
Norway	Svensson
Spain	Nelson-Siegel (before 1995), Svensson
Sweden	Svensson
United Kingdom	Svensson
USA	Smoothing splines

feature is the prevalence of the Nelson-Siegel and Svensson smoothing procedures. Countries that use the Nelson-Siegel and Svensson procedures have a local government bond market that is much thinner than that of the two countries that use polynomial smoothing, the USA and Japan. To understand why Nelson-Siegel and Svensson are such attractive models for forward rate smoothing, especially in thin markets, we will now turn our attention to studying their characteristics.

2.3.1 The Nelson-Siegel and Svensson Forward Rate Models

Both the Nelson-Siegel and Svensson models prescribe a particular functional form to the forward rate curve, whose parameters can then be chosen to fit an observed set of bond prices.

The Nelson-Siegel model is defined by the following formula:

$$f(t,T) = \beta_0 + \beta_1 e^{-\frac{T-t}{\tau}} + \beta_2 \frac{T-t}{\tau} e^{-\frac{T-t}{\tau}} \tag{2.14}$$

while the Svensson model is given by

$$f(t,T) = \beta_0 + \beta_1 e^{-\frac{T-t}{\tau_1}} + \beta_2 \frac{T-t}{\tau_1} e^{-\frac{T-t}{\tau_1}} + \beta_3 \frac{T-t}{\tau_2} e^{-\frac{T-t}{\tau_2}} \tag{2.15}$$

Since the Nelson-Siegel model is a particular case of the Svensson model (when $\beta_3 = 0$), we will focus our attention to the more general version, our conclusions being easily adapted to the simpler case.

The economic justification for the Svensson model is thin; its main attraction is that it allows for a parsimonious representation of the forward rate curve, and that its analytical form is simple. Qualitatively speaking, a Svensson curve consists of a constant value that is modulated by a simple decaying exponential (whose maximum magnitude is reached at $t = 0$), and two terms that impose the existence of one peak[5] (or bump) each. Government bond traders generally think that forward rate curves have a simple underlying shape, and that if a forward rate model exhibits excessive oscillation (low smoothness), then it is not an accurate representation of the reality.

In addition, the Svensson curve imposes an asymptotically constant value for

[5] In this sense a "peak" can be both positive or negative.

the interest rate:

$$\lim_{t\to\infty} f(t,T) = \beta_0 \tag{2.16}$$

Traders know, based on the examination of coupon bond yield curves, that yields of bonds toward the long end manifest a strong tendency toward plateauing (in other words, toward the long end of the yield curve the observed yields increase very slowly, if at all). By extrapolation of this observation, the Svensson model has been designed so that the implied forward rates are asymptotically constant. This avoids many of the problems that polynomial models of the forward rate curve exhibit, namely, that forward rates extrapolated beyond the long end of the forward rate curve tend to grow (or to decrease) without limit.[6] We do point out, however, that an implied asymptotically constant forward rate is not necessarily realistic (longer-term lending likely involves more risk, hence lenders will probably expect more compensation), nor is it necessarily consistent with financial theory. We will return to this point later.

The functional form chosen for the Svensson model also biases forward rates to be non-negative. Pathological behavior could theoretically arise, and certain combinations of parameters could produce negative forward rates, but in our practice we have never seen such a phenomenon.

To illustrate the behavior of the Svensson forward rate curve we will now examine the Svensson forward rate curve fitted for January 6, 1999, and based on the GovPX forward rate data (see figure 2.1). This is a typical day for our dataset, possibly with a little more noise than usual, given that it is early in the calendar year. The parameter values determined for the day under study are given in table 2.2. Unsurprisingly, given the terms of the Svensson curve, the overall shape of

[6]In general, polynomials are notoriously inappropriate for extrapolation.

Table 2.2: Svensson model parameters for January 6, 1999. Smoothing based on GovPX data.

β_0	β_1	β_2	β_3	t_1	t_2
0.057253	-0.013179	0.139560	-0.158461	2.232558	2.588345

the forward rate curve is simple: after starting at approximately 4.4%, the curve has two gentle "bumps," then it asymptotically converges to its limit value of approximately 5.7%. Should the need arise, extrapolation of forward rates beyond the 30-year limit is not problematic, as the asymptotic value is practically reached after 20 years. Overall, the general long term tendency of the curve is for the forward rate to grow, implying increasing, then (almost) plateauing long-term bond yields, in accordance with typical patterns observed in the market.

A big advantage of the Svensson model is that it is not critically affected by the lack of bond pricing information at the long end; its asymptotic behavior almost guarantees a reasonable behavior.[7] This, together with the fact that the model has only six parameters (four for the Nelson-Siegel model), is the major reason for the widespread use of the Svensson model by the national banks of states with a relatively thin national bond market.

The widespread and entrenched use of the Svensson model, and our interest in modeling the forward rate curve with polynomials, raises the question of whether the Svensson curve encodes a more complex behavior than that of a simple polynomial. Recognizing that the examination of this question for one day only is

[7] If t_1 or t_2 are large, and the corresponding betas are not negligible, one can observe Svensson curves whose asymptotic behavior can not emerge by the usual time horizon of 30 years. For our data set, however, this is rarely a problem.

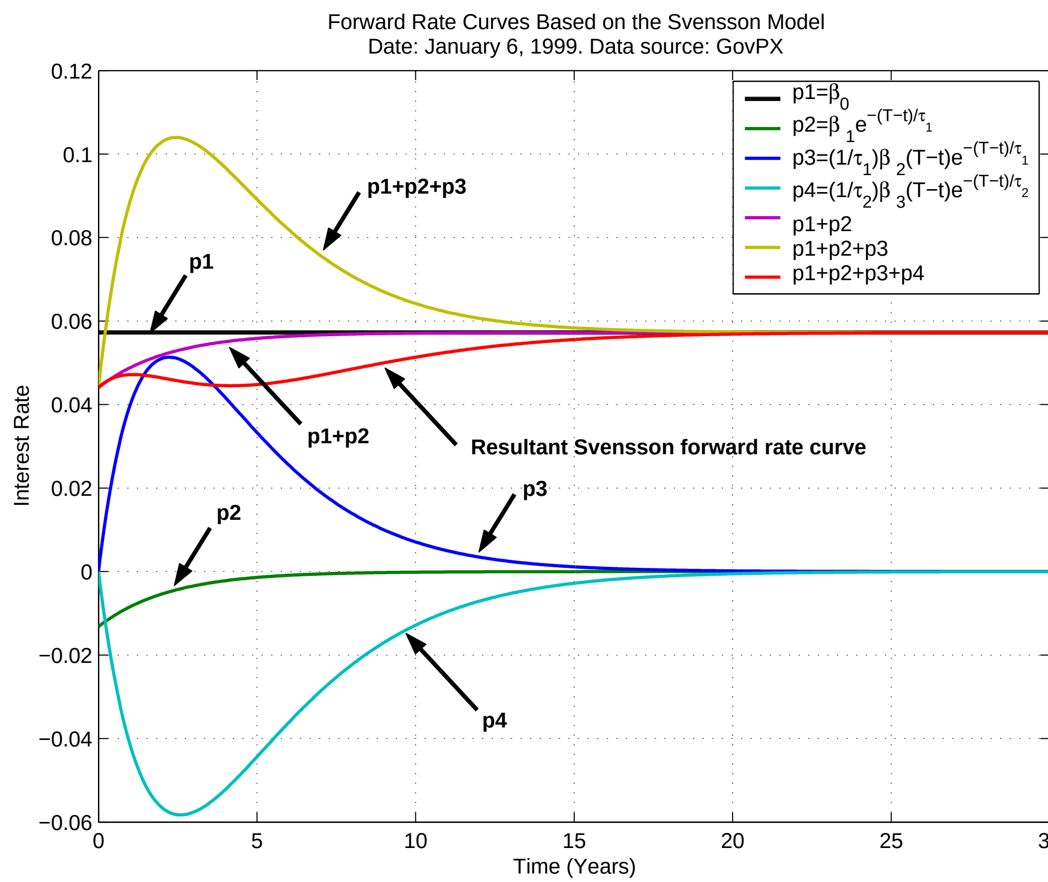

Figure 2.1: Shape of the Svensson forward rate curve as of January 6, 1999, based on GovPX data. The individual terms and their combinations are shown separately, to better illustrate how the shape of the final curve emerges.

inherently limited in the amount of knowledge gained,[8] we will now illustrate the possibility of approximating the Svensson curve by a polynomial on the long end. We will use a fourth degree polynomial, as this is the degree of the polynomials that define the smoothest forward rate curves.

Figure 2.2 shows that a fourth degree polynomial is a reasonable approximation of the Svensson curve toward the long end of the curve. This is not surprising, given the well-known asymptotic behavior of the terms in the Svensson model. We also see that the approximating polynomials exhibit an oscillating behavior around their target curve; this is a feature of polynomial approximations. The qualitative idea we retain from this analysis is that fourth degree polynomials are capable of exhibiting behavior substantially similar to that of the Svensson curve toward the long end of the forward rate curve.

A good description of how the Svensson model is applied at the bank of Canada can be found in (Bolder and Streliski 1999).

2.3.2 Yield Curves, Forward Rates, and the Federal Reserve

Forward rate curves can not be observed directly, and neither can the yield curve based on forward rates ($y_0^{t_B}(t)$). The yield curve corresponding to the coupon bonds, however, can be observed directly, if one has access to Treasury price data. In figure 2.3 we present the distribution of bid and ask yields for June 22, 1999, based on CRSP data, and the distribution of yields as based on the latest intra-

[8]We did study this question more systematically, over the period of March 15, 1999 - July 31, 1999, and the general results agree with our conclusions here. As long as we recognize the limitations of our argument, the insight gained will still be valuable.

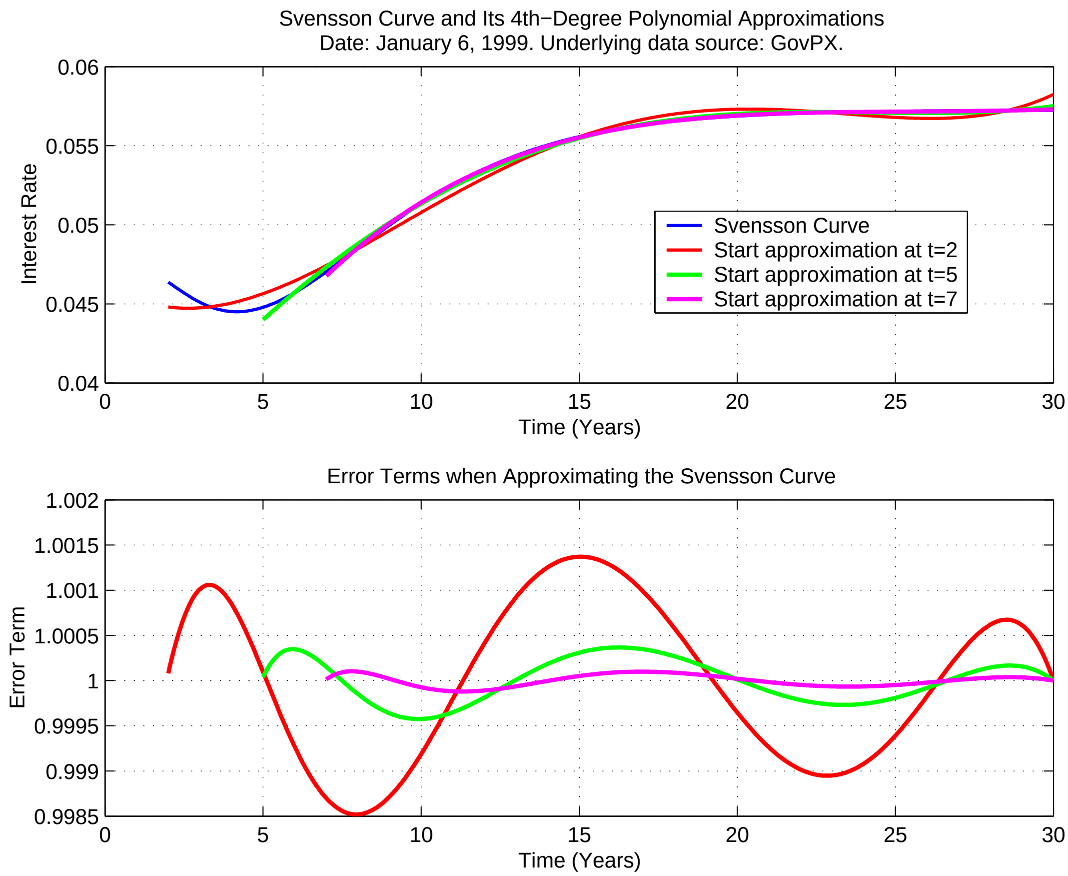

Figure 2.2: The long end of the Svensson curve can reasonably be approximated by using a 4th degree polynomial, even when the domain of the approximation encompasses 28 years. The accuracy of the estimation improves rapidly as the left end of the approximating interval is moved to the right. The bottom graph shows the error term, expressed as the ratio of the Svensson discount factor and that of its three approximations.

day trades recorded by GovPX on the same date. The most remarkable feature of the Treasury yields is that the on-the-run long Treasuries trade at a significantly lower yield (thus, they are relatively more expensive) than off-the-run bonds of comparable maturity. This graph also exhibits a relatively steep initial upward slope, followed by a plateauing and a slight decrease toward the end of the curve.

The domed pattern exhibited by yields between 15 and 30-years is very typical, and it expresses the reduced liquidity that these bonds exhibit. The yields for the on-the-run Treasuries suggest an upward sloping yield curve. If one includes the CRSP data, however, the yield curve seems to achieve a maximum around 20 years, after which it exhibits a gentle decrease. A similar pattern is exhibited by the bond yields between 5 and 10 years.

Many smoothing procedures that are provided in the literature rely on price quotes as their input (see, for example, (Bolder and Streliski 1999) and (Bolder and Gusba 2002)). We conjecture that such an approach is profoundly flawed, primarily because it will tend to bias the end of the smoothed forward rate curves downward to compensate for the decrease in yields toward the longest maturities. Indeed, it is common to see forward rate curves with a negative slope developing close to the end of the curve (Bolder and Streliski 1999, figures 3 and 4)). Without explicitly modeling liquidity effects one should not rely on quote prices for forward rate smoothing (nor, for that matter, for yield curve smoothing).

Alternatively, and this is the approach we take, one could smooth the forward rate curve relying on the liquid instruments at the short end, and only on the liquid on-the-run Treasuries at the long end. Liquidity can then be modeled separately, in a second stage. Market microstructure considerations strongly suggest that the use of bid/ask prices as proxies for true prices of illiquid bonds in smoothing

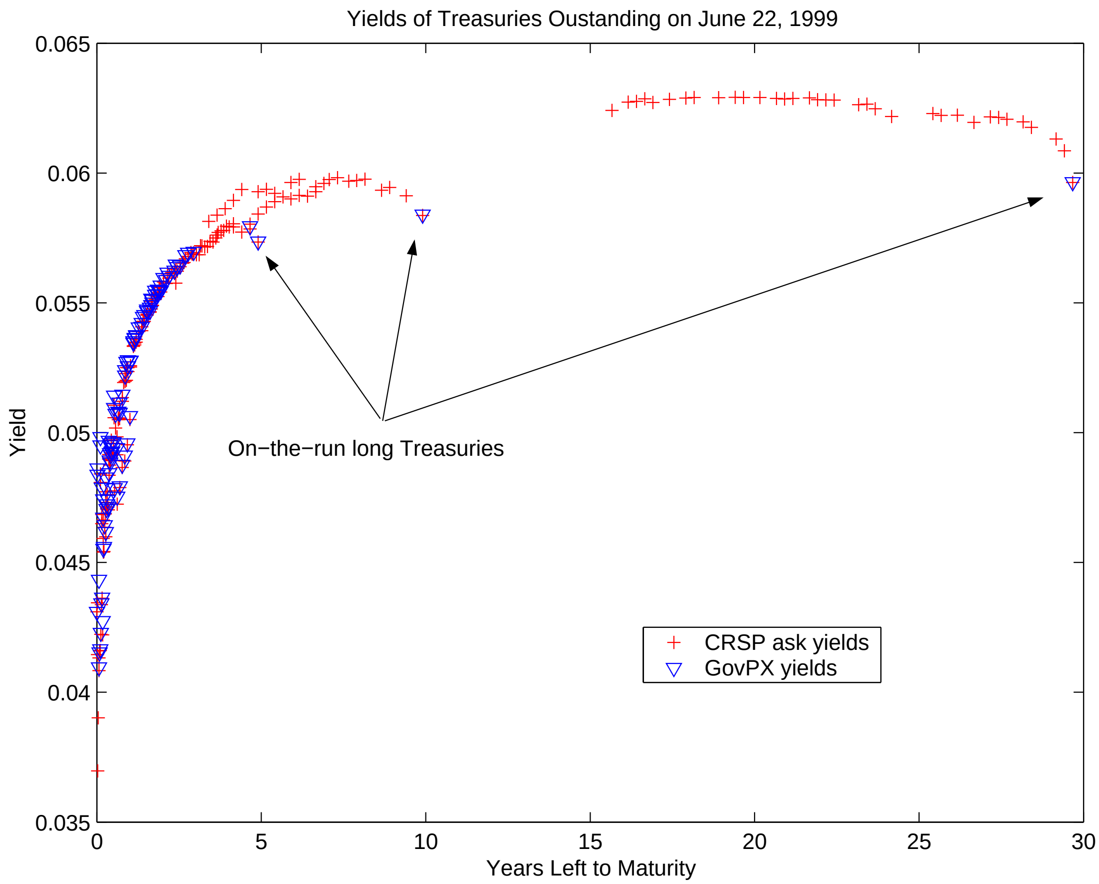

Figure 2.3: A representation of the yield curve of Treasury bills, notes and bonds on June 22, 1999, using pricing data from CRSP and GovPX. Only the CRSP ask yields are represented; at this scale the CRSP bid curve is hard to distinguish from the ask curve, especially at the long end. Note the significantly lower yields of the on-the-run long Treasuries compared to instruments similar in maturity. We argue in the text that there are systematic differences between the yields of on-the-run and off-the-run long term bonds, and that smoothing should ignore illiquid bid/ask prices (or model liquidity explicitly). Because on-the-run Treasury prices are typically present in GovPX data, we can retrieve all the information we need for smoothing from GovPX.

procedures has a distorting effect. Even more than in the case of prices for heavily traded instruments, the sequence of bid/ask prices for illiquid bonds might be, and are, non-contemporaneous. In addition, these bid/ask prices often depend heavily on the quantities traded, since the market is not deep for illiquid instruments. These issues have been studied in depth for equity markets (see, for example, (O'Hara 1997)). There are relatively few studies of liquidity in Treasury markets; for a recent empirical investigation based on GovPX data the reader is directed to (Fleming 2003). Related markets strongly hint that the use of dealer quotes has market distorting effects. In a recent paper on the term structure of commercial paper[9] (Downing and Oliner 2004), for example, two researchers at the Federal Reserve Board found that the use of actual transaction data corrected for known year-end effects supports a short-term version of the expectation hypothesis, while the use of dealer quotes definitely rejects it.[10] Such distortions are even more likely toward the long end of the trading time horizon.

The Federal Reserve publishes neither smoothed forward rates, nor full yield curves. Rather, the Fed publishes daily yields for certain constant maturities (1, 3, 6 months, and 1, 2, 3, 5, 7, 10, 20 years).[11] A yield curve is interpolated with cubic splines that are used to smooth "composites of quotations obtained

[9]Commercial paper is a class of high-quality, short term (typically under 90 days) debt issued by low-risk corporations. We will discuss commercial paper in more detail when we will address arbitrage conditions imposed on our smooth forward rate curves.

[10]We should mention here that the study involved dealer-quotes and transaction data sets that do not overlap in time. Due to this methodological limitation, the conclusions must be treated with caution. Still, this is only one argument that we bring in support of not including dealer quotes in smoothing procedures.

[11]We follow closely the description provided by the Federal Reserve on its financial statistics web pages. For further details, please refer to http://www.treas.gov/offices/domestic-finance/debt-management/interest-rate/yield.html.

by the Federal Reserve Bank of New York" on on-the-run Treasuries. Once a yield curve is obtained, it is used to produce the constant maturity yields that are published. This procedure is complicated by the fact that the market requires long term (i.e. 30-year) yield estimates, which can not be directly provided due to the discontinuation of the 30-year bond in February 2002. The Fed handles this problem by extrapolating the slope of the yield curve at $t = 20$ years to $t = 30$ years.

Following (van Deventer and Imai 1997), we mention that yield curve smoothing has at least two important drawbacks. First, the forward rates inferred from the smoothed yield curve tend to be volatile, especially toward the long end, to such an extent that the forward values implied are implausible. Second, the forward rates implied by the smoothed yield curves are not smooth, having non-differentiable first derivatives. The third reason for rejecting yield curve smoothing is that the forward rate is a fundamental concept, while the yield curve is a derived one. The need to model the forward rate curve directly, as opposed to the yield curve, is the prevalent view of the research body we have surveyed.

2.4 The Ends of Forward Rate Curves

The examination of the short and long end of forward rate curves yields insights that we will find useful in smoothing. We will now proceed with such an examination.

2.4.1 The Short End

At the short end, a lot of information is available about risky, but otherwise very short term interest rates. Examples of such short term rates are those of federal funds, repo rates, and commercial paper rates, which we have addressed previously. These rates can be used to provide upper bounds on the forward rates at the short end. The actual shape of the forward rate, however, is difficult to determine. The main reason for these difficulties is that the relative size of amounts earned as interest for very small periods of time is very small, and can easily be swamped by market distortions caused by low liquidity close to maturity. Also, the interest accumulated over short time horizons is comparable to the gap between two successive price levels.

We illustrate these problems in figure 2.4. Given a short bond with a unique cash flow at maturity c_M, maturity t_M, and yield y, the formula below gives the bond's price:

$$p_B(t) = c_M e^{-y(t)(t_M - t)} \qquad (2.17)$$

From here we immediately get:

$$\Delta y = \frac{1}{t_M - t} \log \frac{1}{1 + \frac{\Delta p}{p_B(t)}} \approx -\frac{f}{t_M - t} \text{ (if } f \text{ is small)} \qquad (2.18)$$

where Δp is the change in price of the bond, and $f = \frac{\Delta p}{p_B(t)}$ is the relative change in price. Thus, for very short maturity bonds a small change in price is amplified approximately in inverse proportion to the time left to maturity. For a Treasury with a leftover maturity of one month, the error is amplified (roughly) 10 times, for an instrument with a week left to maturity, the error is amplified 50 times.

By examining the short end of the curve, some instances of this behavior become apparent. Consider the situation represented in figure 2.5, which shows the yields of the shortest available Treasuries as reported by GovPX. The reader should note how the yields of bills and coupon Treasuries diverge. It is a well-known fact of Treasury markets that notes and bonds of very short maturities exhibit low

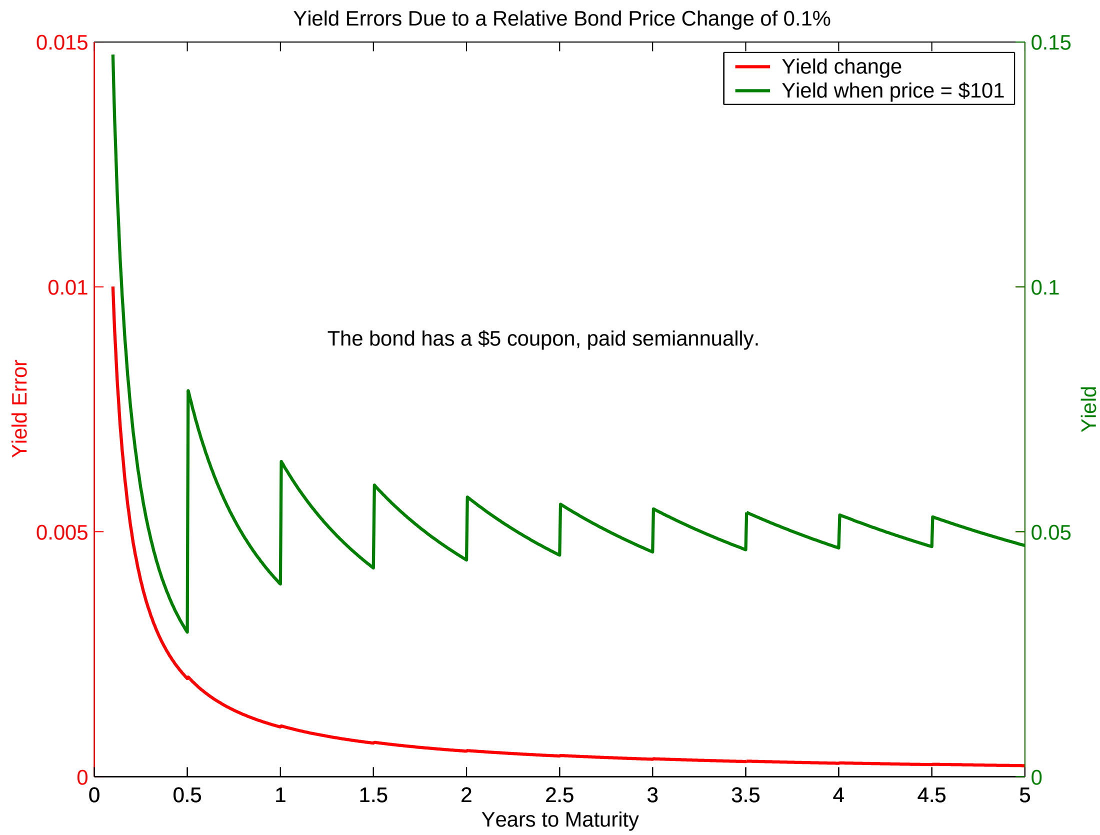

Figure 2.4: A small change in prices can induce a large change in yields for very short bonds. The graph illustrates the change in a bond's yield across time, when the price of the bond is kept constant, at $101/$100 face value. This example has been artificially constructed, no real bonds would exhibit such a constancy of price across long periods of time.

liquidity (they can only be sold at relatively depressed prices), and so they trade at higher yields. If we examine the instruments due to mature in a week shown in the figure, we note that their yield differs by approximately 60 basis points; even the yield of essentially identical bills differs by approximately 15 basis points. While the forward rate curve seems to tend downward, there are hints that at the very end it might turn up.[12]

2.4.2 The Long End

Interest rate information on long term Treasuries, or on other borrowings, is scarce.[13] Fortunately, economic reasoning, and limit properties of various interest rate models provide some support. While there are different theories of what forward rates actually represent (for details see section 2.2), it is accepted that the shape of the forward rate curve is based on, and reflects information contemporaneous with the curve itself. Given the practical limitations of predicting the future, any projections made for 20 or 30 years are inherently very imprecise. This fact is widely recognized. In addition, we posit that the amount of differential information available to distinguish, between, say, a moment in time 29.9 years in the future, and a moment in time 30 years in the future is negligible. Based on

[12]The literature (Cook and LaRoche 1993, pp. 8) mentions a difference of approximately 20-25 basis points between the federal funds rate, and the overnight repo rate. If true, the repo rate for this day would be just under 4.5%, leaving a gap of around 10 basis points between the repo rate and the short rate. Overnight repos are very low-risk, so a 10 basis point gap is much more credible than the gap of over 100 basis points implied by the extrapolation of one week bill yields. Of course, the upturn might be due to liquidity effects only.

[13]Long-term residential and commercial mortgages could be the source of interest rate information at the long end. Unfortunately, mortgages are complicated instruments, as they involve a variety of risks, including default risk and prepayment risk. These factors make mortgage rates impossible to use, in the absence of a reliable valuation theory.

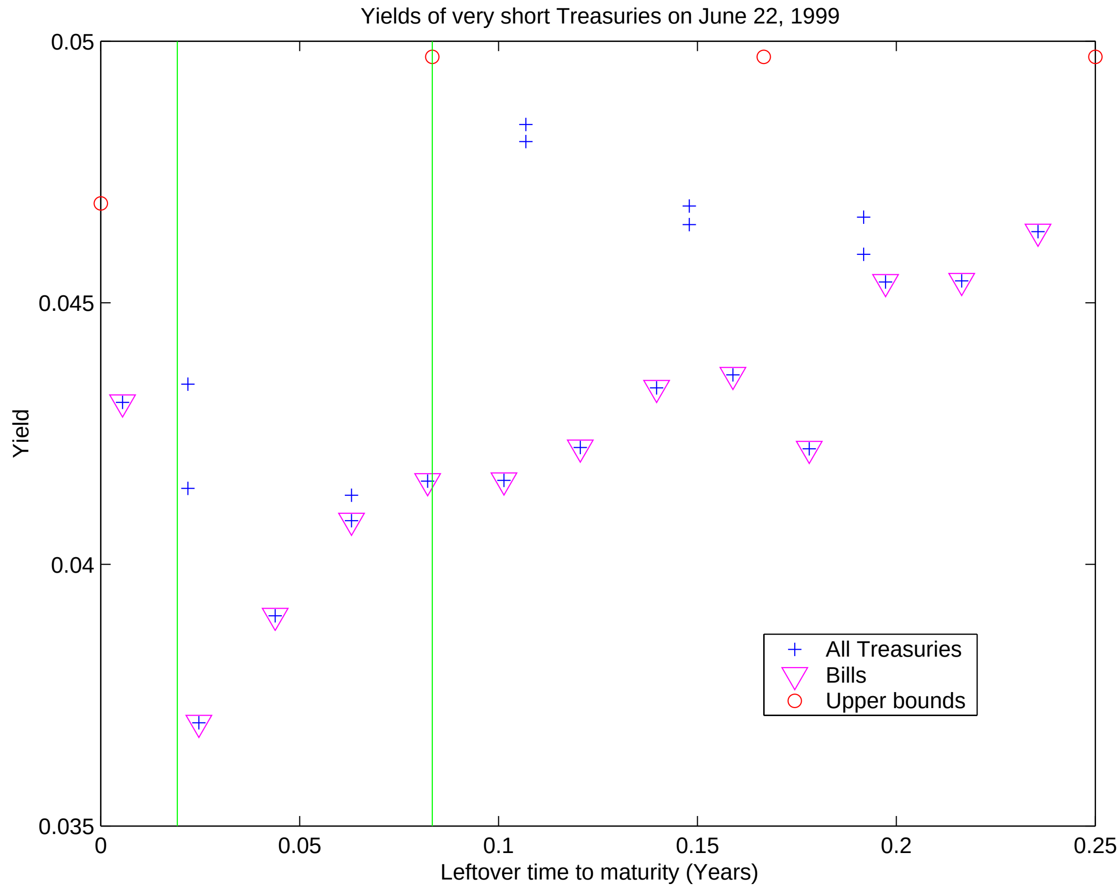

Figure 2.5: The yields of bonds close to the short end exhibit high variability. In this case, the yields of coupon Treasuries diverge from the yields of bills. Vertical bars mark one week, and one month from maturity, respectively. Note the approximately 60 basis point range of yields for instruments due to mature in one week. The federal funds rate and the commercial paper rates are also represented, as they are upper bounds for forward rates. Data source: GovPX.

this observation, it is reasonable to assert that at its extreme rightmost end the curve must be straight.

Looking out to the segment of the curve that precedes its right end, similar, but not identical arguments can be made. Over the longer term it is conceivable that the market embeds differential information with respect to moments of time that are not too close to each other, and this information affects the shape of the curve. To the extent that such differential information exists, however, it is likely that its effect reveals itself slowly, and only over the long term. Consequently, the forward rate should be "smooth." The importance of smoothness, as a feature of forward rate curves, has been recognized previously (see, for example, (van Deventer and Imai 1997), and (Manzano and Blomvall 2004)). Maximum smoothness has been seen previously as an extremal condition that leads to easy mathematical treatment. We posit that maximum smoothness toward the long end of the curve is an unavoidable consequence of the limited differential information that one can project far into the future. The argument above is qualitative and does not determine a lower bound for the smooth portion of the forward rate curve.

Assuming a particular model for the evolution of forward rates, one can discover asymptotic results about the behavior of forward rates. As pointed out in (McCulloch 2000), if transaction costs are included, then future interest rates are indeterminate. Indeed, if we assume that there is a small bid/ask spread of 2ϵ,[14]

[14] For simplicity, we assume that the bid and ask prices are symmetrically distributed around the true price, but this property is not essential for the result to hold.

we can distinguish between bid and ask yields:

$$
\begin{aligned}
\lim_{T\to\infty} y^T_{bid}(t) &= -\lim_{T\to\infty} \frac{1}{T-t}\log(\max(p(t,T)-\epsilon,0)) = \infty \\
\lim_{T\to\infty} y^T_{ask}(t) &= -\lim_{T\to\infty} \frac{1}{T-t}\log(\max(p(t,T),\epsilon)) = 0
\end{aligned}
\tag{2.19}
$$

McCulloch provides a realistic example, where $p(t,T)$ falls under $\epsilon = \frac{2}{32}$ after approximately 123 years, much longer than our 30-year horizon. Due to the existence of trading costs in the market, we can not directly prove that long-term rates will not fall. On the other hand, the study of the forward rate for horizons over 100 years is beyond our scope. Given the limited information, and the profound changes that the world economy will likely undergo in 100 years, it would be too much to expect to develop accurate theories spanning such long intervals. Did any economist in 1904 predict the advent of international air travel, or - to appeal to a well-worn example - the Internet?

Our data extends to 30 years only, and at most we can hope to extrapolate a few years beyond that. In line with the empirical observations that in general interest rates increase in time, we hold that this must be true in most cases at the long end of the forward rate curve, as well as for a reasonable extrapolation period beyond 30 years. Any interest rate model that allows only for forward rates whose growth rate is zero, or very small around the 30-year time horizon is not realistic. In our experience, this is the case with the Svensson curve, whose asymptotic properties tend to impose a very small value of the first derivative close to 30 years, or shortly after that. For June 22, 1999, for example, our Svensson curve fitted to GovPX data has a slope of approximately -0.35 basis points/year.

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